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LETTER TO THE EDITOR

Gauge transformation and bi-Hamiltonian structure of a finite-dimensional integrable system reduced from a soliton equation

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Abstract. We present a method for using a gauge transformation to construct a bi-Hamiltonian structure of a finite-dimensional integrable Hamiltonian system reduced from a soliton equation. This is used to construct the bi-Hamiltonian structure for two systems which are related to the second-order polynomial spectral problem and its modified spectral problem, respectively.

Some finite-dimensional integrable Hamiltonian systems have been shown to possess bi-Hamiltonian structures (see, for example, [1–4]). One common way of constructing a bi-Hamiltonian structure is to use the map between two integrable Hamiltonian systems.

In a number of recent papers [5–8], we developed a straightforward way to obtain a hierarchy of finite-dimensional integrable Hamiltonian systems from a hierarchy of integrable nonlinear evolution equations. By restricting the phase space to the invariant subspace of the recursion operator, a constraint on potential can be found and the associated Lax pair under this constraint become finite-dimensional integrable and commuting Hamiltonian systems. On the other hand, there exist gauge transformations between some spectral problems. Thus it is natural to look for a method of constructing the bi-Hamiltonian structure for these kinds of finite-dimensional integrable Hamiltonian systems by using the gauge transformation. We illustrate these ideas by constructing a bi-Hamiltonian structure for the finite-dimensional integrable Hamiltonian systems related to the second-order polynomial spectral problem and its modified spectral problem, respectively.

For the second-order polynomial spectral problem

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}0 & 1\\ \lambda^m - \Sigma_{i=0}^{m-1}\lambda^i u_i & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{1}$$

the associated evolution equations given in [9] can be rewritten as [8]

$$u_{t_n} = DL^n u \tag{2}$$

where $D=\partial/\partial x$, $DD^{-1}=D^{-1}D=1$, $u=(u_0,\dots,u_{m-1})$,

$$L=\begin{pmatrix}0 & \dots & 0 & J_0\\ 1 & \dots & 0 & J_1\\ \dots & \dots & \dots & \dots\\ 0 & \dots & 1 & J_{m-1}\end{pmatrix}$$

$$J_0=\tfrac{1}{4}D^2+u_0-\tfrac{1}{2}D^{-1}u_{0x} \qquad J_i=u_i-\tfrac{1}{2}D^{-1}u_{ix} \qquad i=1,\dots,m-1.$$

For distinct λ_j, we now consider the following system

$$\phi_{1jx}=\phi_{2j}$$
$$\phi_{2jx}=\left(\lambda_j^m-\sum_{i=0}^{m-1}\lambda_j^i u_i\right)\phi_{1j}\qquad j=1,\ldots,N. \tag{3}$$

By restricting u to the invariant subspace of the recursion operator L, a constraint on u is found as follows in [8]

$$u_0=\langle\Phi_1,\Phi_1\rangle+C \tag{4a}$$

$$u_{m-k}=\sum_{i=1}^{k}(-1)^{i-1}\frac{(i+1)}{2^i}\sum_{l_1+\ldots+l_i=k-i}\langle\Lambda^{l_1}\Phi_1,\Phi_1\rangle\ldots\langle\Lambda^{l_i}\Phi_1,\Phi_1\rangle$$
$$k=1,\ldots,m-1 \tag{4b}$$

where C is a constant; hereafter $l_1\geqslant 0,\ldots,l_i\geqslant 0$, $\Phi_1=(\phi_{11},\ldots,\phi_{1N})^T$, $\Phi_2=(\phi_{21},\ldots,\phi_{2N})^T$, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$, $\langle\cdot,\cdot\rangle$ is the inner product in $\boldsymbol{R}^N$. Under this constraint on u, (3) becomes a finite-dimensional integrable Hamiltonian system [8]

$$\phi_{1jx}=\frac{\partial H_1}{\partial\phi_{2j}}\qquad \phi_{2jx}=-\frac{\partial H_1}{\partial\Phi_1} \tag{5a}$$

with

$$H_1=\tfrac{1}{2}\langle\Phi_2,\Phi_2\rangle+\sum_{i=0}^{m}(-\tfrac{1}{2})^{i+1}\sum_{l_1+\ldots+l_{i+1}=m-i}\langle\Lambda^{l_1}\Phi_1,\Phi_1\rangle\ldots\langle\Lambda^{l_{i+1}}\Phi_1,\Phi_1\rangle+\tfrac{1}{2}C\langle\Phi_1,\Phi_1\rangle. \tag{5b}$$

The integrals of the motion in involution for $(5a)$ are given by

$$F_k=\tfrac{1}{2}\langle\Lambda^{k-1}\Phi_2,\Phi_2\rangle+\sum_{i=0}^{m}(-\tfrac{1}{2})^{i+1}\sum_{l_1+\ldots+l_{i+1}=m-i}\langle\Lambda^{l_1}\Phi_1,\Phi_1\rangle\ldots\langle\Lambda^{l_i}\Phi_1,\Phi_1\rangle$$
$$\times\langle\Lambda^{l_{i+1}+k-1}\Phi_1,\Phi_1\rangle+\tfrac{1}{4}\sum_{j=0}^{k-2}[\langle\Lambda^j\Phi_1,\Phi_1\rangle\langle\Lambda^{k-2-j}\Phi_2,\Phi_2\rangle$$
$$-\langle\Lambda^j\Phi_1,\Phi_2\rangle\langle\Lambda^{k-2-j}\Phi_1,\Phi_2\rangle]+\tfrac{1}{2}C\langle\Lambda^{k-1}\Phi_1,\Phi_1\rangle\qquad k=1,2,\ldots. \tag{6}$$

The modified spectral problem for (1) is of the form [9]

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}v_0 & \lambda\\ \lambda^{m-1}-\Sigma_{i=1}^{m-1}\lambda^{i-1}v_i & -v_0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{7}$$

with $v=(v_0,\ldots,v_{m-1})$,

$$u_0=-v_{0x}-v_0^2\qquad v_i=u_i\qquad i=1,\ldots,m-1. \tag{8}$$

From (2) and the Miura transformation (8), it is easy to obtain the evolution equations associated with (7):

$$v_{t_n}=D\tilde{L}^n v \tag{9}$$

with

$$\tilde{L}=\begin{pmatrix} 0 & 0 & \dots & 0 & \tilde{J}_0 \\ -D-2D^{-1}v_0D & 0 & \dots & 0 & \tilde{J}_1 \\ 0 & 1 & \dots & 0 & \tilde{J}_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \tilde{J}_{m-1} \end{pmatrix}$$

$$\tilde{J}_0=-\tfrac{1}{4}D+\tfrac{1}{2}v_0 \qquad \tilde{J}_i=v_i-\tfrac{1}{2}D^{-1}v_{ix} \qquad i=1,\dots,m-1.$$

Throughout the letter no boundary condition on u and v is imposed. Firstly we want to reduce an integrable Hamiltonian system from (7) by using the method in [6].

Consider the system

$$\begin{aligned} \psi_{1jx}&=v_0\psi_{1j}+\lambda_j\psi_{2j} \\ \psi_{2jx}&=\lambda_j^{m-1}\psi_{1j}-\sum_{i=1}^{m-1}\lambda_j^{i-1}v_i\psi_{1j}-v_0\psi_{2j} \qquad j=1,\dots,N. \end{aligned} \tag{10}$$

It is easy to verify that if ψ_{1j} and ψ_{2j} satisfy (10), we have

$$\tilde{L}A_j=\lambda_jA_j+\sum_{i=2}^{m}\beta_i^{(j)}e_i \qquad j=1,\dots,N \tag{11}$$

where $\beta_i^{(j)}$ are integral constants, $e_1=(1,0,\dots,0)^T,\dots,e_m=(0,\dots,0,1)^T$,

$$\begin{aligned} A_j&=(A_{j1},\dots,A_{jm})^T \\ A_{j1}&=-\tfrac{1}{2}\psi_{1j}\psi_{2j} \\ A_{jk}&=\lambda_j^{m-k}\psi_{1j}^2-\sum_{i=0}^{m-k-1}\lambda_j^i\tilde{J}_{k+1}\psi_{1j}^2 \qquad k=2,\dots,m-1 \\ A_{jm}&=\psi_{1j}^2. \end{aligned}$$

Notice that

$$\tilde{L}\sum_{i=1}^{m}\beta_i^{(j)}e_i=\sum_{i=2}^{m-1}\beta_i^{(j)}e_{i+1}+\tfrac{1}{2}\beta_m^{(j)}. \tag{12}$$

If we take (C_1 is a constant)

$$v=\sum_{j=1}^{N}A_j+C_1e_1 \tag{13}$$

we find from (11) and (12) that the linear space spanned by $\{A_1,\dots,A_N,e_1,\dots,e_m\}$ is an invariant subspace of $\tilde{L}$. This property enables us to obtain the results for system (10) analogous to the theorems in [8]. However, we omit them here.

In a similar manner to [8], (13) leads to the following constraint on v:

$$v_0=-\tfrac{1}{2}\langle\Psi_1,\Psi_2\rangle+C_1 \tag{14a}$$

$$v_{m-k}=\sum_{i=1}^{k}(-1)^{i-1}\frac{(i+1)}{2^i}\sum_{l_1+\dots+l_i=k-i}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\dots\langle\Lambda^{l_i}\Psi_1,\Psi_1\rangle$$

$$k=1,\dots,m-1 \tag{14b}$$

where $\Psi_1=(\psi_{11},\ldots,\psi_{1N})^T$, $\Psi_2=(\psi_{21},\ldots,\psi_{2N})^T$. Under this constraint on v, (10) becomes a Hamiltonian system

$$\psi_{1jx}=\frac{\partial\tilde{H}_1}{\partial\psi_{2j}} \qquad \psi_{2jx}=-\frac{\partial\tilde{H}_1}{\partial\psi_{1j}} \tag{15a}$$

with

$$\tilde{H}_1=-\tfrac{1}{4}\langle\Psi_1,\Psi_2\rangle^2+C_1\langle\Psi_1,\Psi_2\rangle+\tfrac{1}{2}\langle\Lambda\Psi_2,\Psi_2\rangle + \sum_{i=1}^{m}(-\tfrac{1}{2})^i\sum_{l_1+\ldots+l_i=m-i}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\ldots\langle\Lambda^{l_i}\Psi_1,\Psi_1\rangle. \tag{15b}$$

We will show later that (15) is an integrable Hamiltonian system.

We find that the gauge transformation

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -\lambda_j^{-1}v_0 & \lambda_j^{-1}\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix} \tag{16}$$

transforms the modified spectral problem (10) into the spectral problem (3) for fixed j. Under the constraint (14), (16) reads

$$\phi_{1j}=\psi_{1j} \tag{17a}$$

$$\phi_{2j}=\lambda_j\psi_{2j}-\tfrac{1}{2}\langle\Psi_1,\Psi_2\rangle\psi_{1j}+C_1\psi_{1j} \tag{17b}$$

which together with (4a) and (8) leads to

$$C=\tilde{H}_1-C_1^2. \tag{17c}$$

Since $\tilde{H}_1$ is constant of motion for (15), (17c) does not contradict the fact that C is a constant. This implies that the transformation $(\Phi_1,\Phi_2,C)=M(\Psi_1,\Psi_2,C_1)$ defined by (17) gives a map between integrable Hamiltonian systems (5) and (15). Following [3, 4], we must extend the phase space to include the constants C and C_1. Using the notation $\Phi^T=(\Phi_1^T,\Phi_2^T,C)$, $\Psi^T=(\Psi_1^T,\Psi_2^T,C_1)$, the systems (5) and (15) are similarly extended as follows, respectively:

$$\Phi_x=B_1\frac{\partial H_1}{\partial\Phi} \tag{5a$'$}$$

$$\Psi_x=\tilde{B}_1\frac{\partial\tilde{H}_1}{\partial\Psi} \tag{15a$'$}$$

where

$$B_1=\tilde{B}_1=\begin{pmatrix}0 & I & 0\\ -I & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Then the Jacobian M' of the map M can be used to construct the second Hamiltonian structure B_2 for the system (5$'$) in the standard way

$$B_2=M'\tilde{B}_1M'^T=\begin{pmatrix}0 & \Lambda-\tfrac{1}{2}\Phi_1\otimes\Phi_1 & \Phi_2\\ -\Lambda+\tfrac{1}{2}\Phi_1\otimes\Phi_1 & \tfrac{1}{2}\Phi_2\otimes\Phi_1-\tfrac{1}{2}\Phi_1\otimes\Phi_2 & -\partial H_1/\partial\Phi_1\\ -\Phi_2^T & (\partial H_1/\partial\Phi_1)^T & 0\end{pmatrix}$$

$$H_0=(\tilde{H}_1-C_1^2)\circ M^{-1}=C$$

where $\otimes$ denotes tensor product. So the system (5) becomes bi-Hamiltonian:

$$\Phi_x = B_1 \frac{\partial H_1}{\partial \Phi} = B_2 \frac{\partial H_0}{\partial \Phi}. \tag{18}$$

As a special case, when taking $m=1$ for (1), (18) is just the bi-Hamiltonian structure for the Garnier system given in [4].

The chain equation

$$B_1 \frac{\partial H_{k+1}}{\partial \Phi} = B_2 \frac{\partial H_k}{\partial \Phi} \tag{19}$$

provides an alternative way to generate the integrals of motion for (5). Indeed, starting with $H_0 = C$, we find that H_k generated by (19) are just F_k given by (6). Since C is the Casimir of B_1, it is easy to show from (19) that F_k are in involution.

Finally we turn to system (15). By substituting (17) into (6), we obtain the integrals of motion for system (15):

$$\begin{aligned}\tilde{F}_{k+1} = {}& \tfrac{1}{2}\langle \Lambda^{k+1}\Psi_2, \Psi_2\rangle - \tfrac{1}{4}\langle \Psi_2, \Psi_2\rangle\langle \Lambda^k \Psi_1, \Psi_1\rangle + C_1\langle \Lambda^k \Psi_1, \Psi_2\rangle \\ & \times \tfrac{1}{4}\sum_{j=0}^{k} [\langle \Lambda^j \Psi_1, \Psi_1\rangle\langle \Lambda^{k-j}\Psi_2, \Psi_2\rangle - \langle \Lambda^j \Psi_1, \Psi_2\rangle\langle \Lambda^{k-j}\Psi_1, \Psi_2\rangle] \\ & + \sum_{i=1}^{m} (-\tfrac{1}{2})^i \sum_{l_1+\ldots+l_i=m-i} \langle \Lambda^{l_1}\Psi_1, \Psi_1\rangle \ldots \langle \Lambda^{l_{i-1}}\Psi_1, \Psi_1\rangle\langle \Lambda^{l_i+k}\Psi_1, \Psi_1\rangle \\ & k = 0, 1, \ldots. \end{aligned} \tag{20}$$

Since the map (17) is inverse, we can compute a second Hamiltonian structure for system (15). Notice that the Jacobian of an inverse function is just the inverse of the Jacobian of the map; we have

$$\begin{aligned}\tilde{B}_0 &= (M')^{-1}B_1((M')^{-1})^T \\ &= \begin{pmatrix} 0 & \Lambda^{-1} - (1/2C_1)\Psi_2 \otimes \Lambda^{-1}\Psi_1 & \partial \tilde{H}_0/\partial \Psi_2 \\ -\Lambda^{-1} + (1/2C_1)\Lambda^{-1}\Psi_1 \otimes \Psi_2 & E & -\partial \tilde{H}_0/\partial \Psi_1 \\ -(\partial \tilde{H}_0/\partial \Psi_2)^T & (\partial \tilde{H}_0/\partial \Psi_1)^T & 0 \end{pmatrix} \\ \tilde{H}_2 &= H_1 \circ M = \tilde{F}_2 \end{aligned} \tag{21}$$

where

$$\begin{aligned} E = {}& \frac{1}{C_1}\sum_{i=1}^{m-1} (-\tfrac{1}{2})^i i \sum_{l_1+\ldots+l_i=m-i-1} \langle \Lambda^{l_1}\Psi_1, \Psi_1\rangle \ldots \langle \Lambda^{l_{i-1}}\Psi_1, \Psi_1\rangle \\ & \times [(\Lambda^{l_i}\Psi_1)\otimes(\Lambda^{-1}\Psi_1) - (\Lambda^{-1}\Psi_1)\otimes(\Lambda^{l_i}\Psi_1)] \end{aligned}$$

$$\begin{aligned} \tilde{H}_0 = {}& \frac{1}{4C_1}\langle \Psi_2, \Psi_2\rangle - \frac{1}{8C_1}\langle \Psi_2, \Psi_2\rangle\langle \Lambda^{-1}\Psi_1, \Psi_1\rangle + \tfrac{1}{2}\langle \Lambda^{-1}\Psi_1, \Psi_2\rangle \\ & + \frac{1}{2C_1}(1 - \tfrac{1}{2}\langle \Lambda^{-1}\Psi_1, \Psi_1\rangle) \sum_{i=1}^{m-1} (-\tfrac{1}{2})^i \\ & \times \sum_{l_1+\ldots+l_i=m-i-1} \langle \Lambda^{l_1}\Psi_1, \Psi_1\rangle \ldots \langle \Lambda^{l_i}\Psi_1, \Psi_1\rangle. \end{aligned}$$

Then (15) becomes the bi-Hamiltonian system

$$\Psi_x = \tilde{B}_1 \frac{\partial \tilde{H}_1}{\partial \Psi} = \tilde{B}_0 \frac{\partial \tilde{H}_2}{\partial \Psi}. \tag{22}$$

Using the chain equation

$$\tilde{B}_1 \frac{\partial}{\partial \Psi} \tilde{H}_k = \tilde{B}_0 \frac{\partial}{\partial \Psi} \tilde{H}_{k+1} \tag{23}$$

and starting with $\tilde{H}_1 = \tilde{F}_1$, we find $\tilde{H}_k = \tilde{F}_k$. Since $(\tilde{H}_1 - C_1^2)$ is the Casimir of $\tilde{B}_0$, (23) can be used to show that $\tilde{F}_k$ are in involution. Distinct λ_j guarantees that $\tilde{F}_1, \ldots, \tilde{F}_N$ are functionally independent. Hence (15) is completely integrable Hamiltonian system in the sense of Liouville.

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